



# Junctions between three-dimensional and two-dimensional non linearly elastic structures

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**JUNCTIONS BETWEEN  
THREE - DIMENSIONAL AND  
TWO - DIMENSIONAL NON  
LINEARLY ELASTIC STRUCTURES**

**Martial AUFRANC**

**Mai 1989**



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**JUNCTIONS BETWEEN THREE - DIMENSIONAL  
AND TWO - DIMENSIONAL NON LINEARLY ELASTIC STRUCTURES.**

**Martial AUFRANC**

**Abstract :**

A mathematical modeling of the junction between three-dimensional and two-dimensional linearly elastic structures has been recently proposed by P.G. CIARLET, H. LE DRET and R.NZENGWA. Their approach is generalized here to non linearly elastic structures.

**RACCORDS DE STRUCTURES BIDIMENSIONNELLES  
ET TRIDIMENSIONNELLES NON LINEAIREMENT ELASTIQUES**

**Résumé :**

P.G. CIARLET, H. LE DRET et R. NZENGWA ont récemment proposé une modélisation de la jonction entre une structure tridimensionnelle et une plaque linéairement élastiques. Nous généralisons ici cette étude au cas de matériaux non linéairement élastiques.

## 1. Introduction :

Junctions between three-dimensional and two-dimensional linearly elastic structures have recently been studied by CIARLET, LE DRET & NZENGWA [1]. They established the  $H^1$ -convergence, as the thickness of the plate approaches zero, of the components of the displacement vector field towards the solution of a limit problem posed simultaneously over the middle surface of the plate and over the three-dimensional open set with a slit. For a numerical study of this limit problem, see AUFRANC [2]. CIARLET & LE DRET [3] [4] justify similarly the boundary conditions of a clamped plate.

In this paper we examine the case of general non-linearly elastic structures : the materials are supposed to be homogeneous and isotropic. For this, the asymptotic expansion method is used and only formal calculations are possible since no results of convergence for this non-linear case exist at present. A similar study has been carried out by CIARLET & DESTUYNDER [5], CIARLET [6], DAVET [7] and RAOULT [8] for a single plate - and most of their results will be used here -, CIMETIERE, GEYMONAT, LE DRET, RAOULT & TUTTEK [9] for rods, CIARLET & PAUMIER [10] for shallow shells; for another point of view about junctions between three-dimensional structures and shallow shells, see CIARLET & MIARA [11].

## 2. The three-dimensional problem :

Let :

$\omega = \{(x_1, x_2) \in \mathbb{R}^2, 0 < x_1 < b_1, |x_2| < a_2\}$ , the middle plane of the plate ;

$\Omega^\epsilon = \omega \times ]-\epsilon, \epsilon[$ , the plate ;

$\gamma_0 = \{(b_1, x_2) \in \mathbb{R}^2; |x_2| \leq a_2\}$  ;

$\Gamma_0^\epsilon = \gamma_0 \times ]-\epsilon, \epsilon[$  ;

$\omega_\beta = \{(x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < \beta, |x_2| < a_2\}$ , the middle plane of the inserted part of the plate ;

$\Omega_\beta^\epsilon = \omega_\beta \times ]-\epsilon, \epsilon[$ , the inserted part of the plate ;

$\mathcal{Q} = \{(x_1, x_2, x_3) \in \mathbb{R}^3; -a_1 < x_1 < \beta, |x_2| < a_2, -a_3 < x_3 < b_3\}$  ;

$\mathcal{Q}_\beta^\epsilon = \mathcal{Q} \setminus \Omega_\beta^\epsilon$ , the three-dimensional part without the inserted part of the plate ;

$\mathcal{L}^\epsilon = \mathcal{Q} \cup \Omega^\epsilon$ , the complete shape ;

$a_1, b_1, a_2, a_3, b_3, \beta$  are positive constants ; we assume that  $\beta < b_1$ .

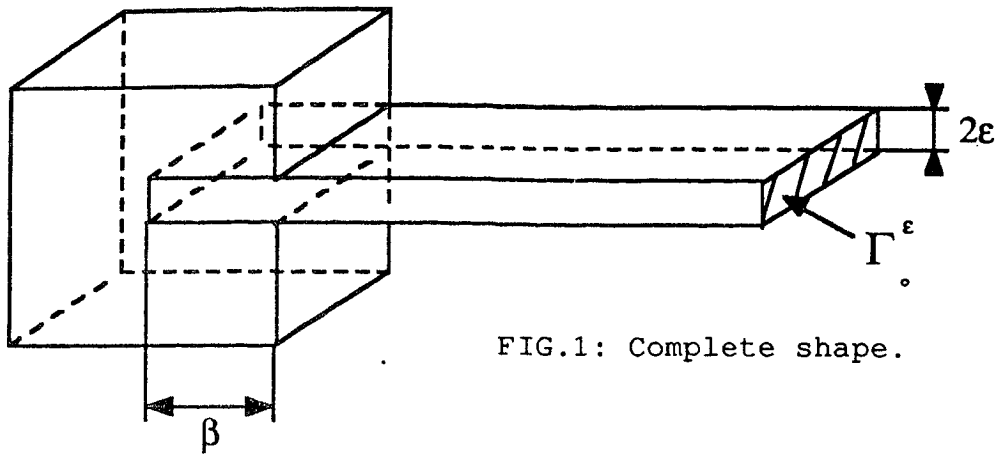


FIG.1: Complete shape.

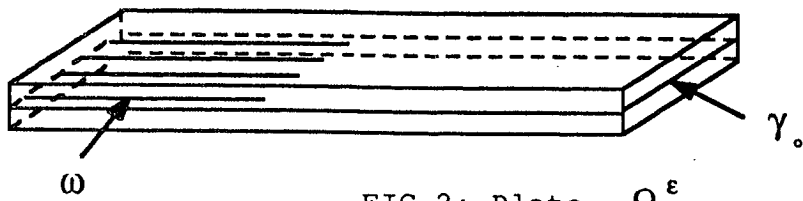


FIG.2: Plate,  $\Omega^\varepsilon$ .

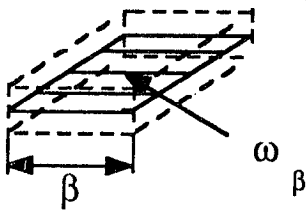


FIG.3: Inserted part of the plate,  $\Omega_\beta^\varepsilon$ .

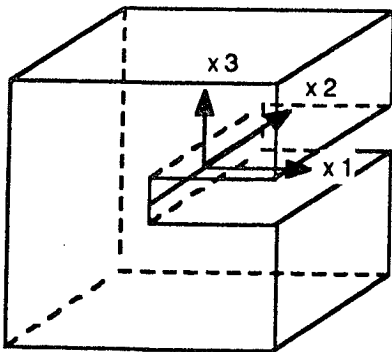


FIG.4: Three-dimensional part without the inserted part of the plate,  $\mathcal{C}_\beta^\varepsilon$ .

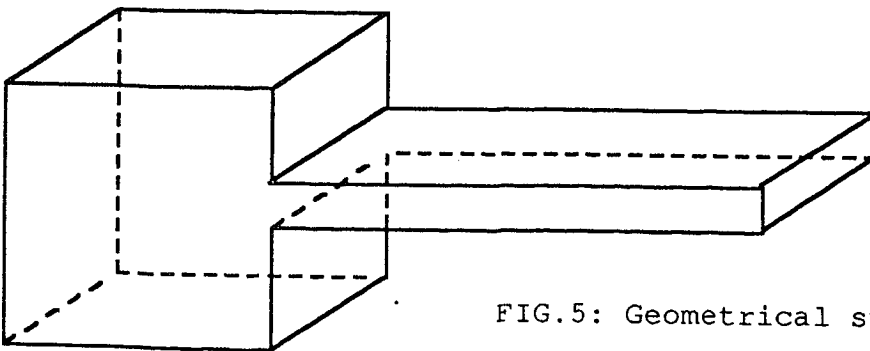


FIG.5: Geometrical support,  $\mathcal{L}^\varepsilon$ .

Hence,  $\mathcal{Q}_\beta^\epsilon$  is a rectangular parallelepiped with a slit  $\Omega_\beta^\epsilon$  in which is inserted a plate  $\Omega^\epsilon$  of middle surface  $\omega$ , with thickness  $2\epsilon$ , as shown by figures 1 to 5.

Since  $\epsilon$  is to be understood as a dimensionless parameter, the thickness of the plate should be written as  $2\epsilon h$ , for some fixed length  $h > 0$ . For simplicity, we assume here that  $h = 1$ .

We denote by  $x^\epsilon = (x_i^\epsilon)$  a generic point in the set  $\mathcal{J}^\epsilon$ ,  $\partial_i^\epsilon$  the partial derivative with respect to  $x_i^\epsilon$ , and by :

$$u^\epsilon = (u_i^\epsilon) : \mathcal{J}^\epsilon \rightarrow \mathbb{R}^3$$

the displacement vector field ; we assume that :  $u^\epsilon = 0$  on  $\Gamma_0^\epsilon$ .

The usual convention for summation is used.

For a displacement vector field  $v : C(v) = (I + \nabla v)^T (I + \nabla v)$  is the right Cauchy-Green strain tensor,  $E(v) = \frac{1}{2} (\nabla v^T + \nabla v + \nabla v^T \nabla v)$  is the Green-Saint Venant strain tensor, and  $e(v) = \frac{1}{2} (\nabla v^T + \nabla v)$  is the linearized strain tensor, where :  $\nabla v = (\partial_i v_j)$ .

We denote by  $\Sigma^\epsilon$  the second Piola-Kirchhoff stress tensor. According to the Rivlin-Ericksen theorem - see for example CIARLET [12] -, the constitutive laws for each material take the form :

$$(1) \quad \Sigma_{ij}^\epsilon = \tilde{\gamma}_1 (\iota_C^\epsilon) \delta_{ij} + \tilde{\gamma}_2 (\iota_C^\epsilon) C_{ij}^\epsilon + \tilde{\gamma}_3 (\iota_C^\epsilon) C_{ik}^\epsilon C_{kj}^\epsilon \text{ in } \mathcal{Q}_\beta^\epsilon,$$

$$(2) \quad \Sigma_{ij}^\epsilon = \gamma_1^\epsilon (\iota_C^\epsilon) \delta_{ij} + \gamma_2^\epsilon (\iota_C^\epsilon) C_{ij}^\epsilon + \gamma_3^\epsilon (\iota_C^\epsilon) C_{ik}^\epsilon C_{kj}^\epsilon \text{ in } \Omega^\epsilon,$$

where  $\tilde{\gamma}_i$  and  $\gamma_i^\epsilon$  are real smooth functions defined on  $]0, +\infty[^3$ , where :

$\iota_C^\epsilon = (I_C^\epsilon, II_C^\epsilon, III_C^\epsilon)$ , with :

$$I_C^\epsilon = \text{tr } (C^\epsilon),$$

$$II_C^\epsilon = \text{tr } (\text{Cof } C^\epsilon)$$

$$III_C^\epsilon = \det C^\epsilon,$$

and where the components of the right Cauchy-Green tensor  $C^\epsilon$  read :

$$(3) \quad C_{ij}^\epsilon = \delta_{ij} + \partial_i^\epsilon u_j^\epsilon + \partial_j^\epsilon u_i^\epsilon + \partial_i^\epsilon u_k^\epsilon \partial_j^\epsilon u_k^\epsilon \text{ in } \mathcal{J}^\epsilon.$$

We assume that the reference configuration is a natural state, i.e. :

$$(4) \quad \sum_{i=1}^3 \gamma_i^\epsilon (3,3,1) = \sum_{i=1}^3 \tilde{\gamma}_i (3,3,1) = 0.$$

Let us recall that the Lamé moduli  $\tilde{\lambda}$ ,  $\tilde{\mu}$  for the three-dimensional part and  $\lambda^\epsilon$ ,  $\mu^\epsilon$  for the plate are :

$$(5) \quad \begin{cases} \tilde{\lambda} = \sum_{i=1}^3 (2 \partial_1 \tilde{\gamma}_i (3,3,1) + 4 \partial_2 \tilde{\gamma}_i (3,3,1) + 2 \partial_3 \tilde{\gamma}_i (3,3,1)), \\ \tilde{\mu} = \tilde{\gamma}_2 (3,3,1) + 2 \tilde{\gamma}_3 (3,3,1), \\ \lambda^\epsilon = \sum_{i=1}^3 (2 \partial_1 \gamma_i^\epsilon (3,3,1) + 4 \partial_2 \gamma_i^\epsilon (3,3,1) + 2 \partial_3 \gamma_i^\epsilon (3,3,1)), \\ \mu^\epsilon = \gamma_2^\epsilon (3,3,1) + 2 \gamma_3^\epsilon (3,3,1). \end{cases}$$

Finally,  $(f_i^\epsilon) \in L^2(\mathcal{J}^\epsilon)$  denotes the given applied body force density. For the sake of simplicity, applied surface force densities are assumed to be zero.

The equilibrium equations for the structure  $\mathcal{J}^\epsilon$  under our assumptions are :

$$(6) \quad - \partial_j^\epsilon (\Sigma_{ij}^\epsilon + \Sigma_{kj}^\epsilon \partial_k^\epsilon u_i^\epsilon) = f_i^\epsilon \quad \text{in } \mathcal{Q}_\beta^\epsilon,$$

$$(7) \quad - \partial_i^\epsilon (\Sigma_{ij}^\epsilon + \Sigma_{kj}^\epsilon \partial_k^\epsilon u_i^\epsilon) = f_i^\epsilon \quad \text{in } \Omega^\epsilon,$$

$$(8) \quad (\Sigma_{ij}^\epsilon + \Sigma_{kj}^\epsilon \partial_k^\epsilon u_i^\epsilon) \tilde{n}_j^\epsilon = 0 \quad \text{on } \partial \mathcal{Q}_\beta^\epsilon \setminus \partial \Omega^\epsilon,$$

$$(9) \quad (\Sigma_{ij}^\epsilon + \Sigma_{kj}^\epsilon \partial_k^\epsilon u_i^\epsilon) n_j^\epsilon = 0 \quad \text{on } \partial \Omega^\epsilon \setminus \partial \mathcal{Q}_\beta^\epsilon,$$

$$(10) \quad (\Sigma_{ij}^\epsilon + \Sigma_{kj}^\epsilon \partial_k^\epsilon u_i^\epsilon) \tilde{n}_j^\epsilon + (\Sigma_{ij}^\epsilon + \Sigma_{kj}^\epsilon \partial_k^\epsilon u_i^\epsilon) n_j^\epsilon = 0 \quad \text{on } \partial \Omega^\epsilon \cap \partial \mathcal{Q}_\beta^\epsilon,$$

where  $\tilde{n}^\epsilon$  and  $n^\epsilon$  denote respectively the unit outer normal vectors along boundaries of the sets  $\mathcal{Q}_\beta^\epsilon$  and  $\Omega^\epsilon$ . Moreover, the boundary and transmission conditions to be satisfied by the displacement are :

$$(11) \quad u^\epsilon = 0 \quad \text{on } \Gamma_0^\epsilon,$$

$$(12) \quad (\text{tr } u^\epsilon |_{\mathcal{Q}_\beta^\epsilon})|_{\partial \mathcal{Q}_\beta^\epsilon \cap \partial \Omega^\epsilon} = (\text{tr } u^\epsilon |_{\Omega^\epsilon})|_{\partial \mathcal{Q}_\beta^\epsilon \cap \partial \Omega^\epsilon}.$$

These equations have to be complemented by the constitutive relation (1) and (2). It should be noted that no existence result seems to be known for the above system.

We now introduce a weak form of these problems  $\mathcal{P}_\epsilon$  defined by (1) to (12).

Let us define :

$$V^\epsilon = \{v = (v_i) \in W^{1,p}(\mathcal{J}^\epsilon)^3 ; v = 0 \text{ on } \Gamma_0^\epsilon\}, \text{ for } p \geq 4 ;$$

$$(L_s^2)^\epsilon = \{\tau = (\tau_{ij}) \in L^2(\mathcal{J}^\epsilon)^9 ; \tau_{ij} = \tau_{ji}\}.$$

The weak form of  $\mathcal{P}_\epsilon$  is described by the variational equations :

$$(13) \quad \forall \tau^\epsilon \in (L_s^2)^\epsilon :$$

$$\begin{aligned} & \int_{\mathcal{Q}_\beta^\epsilon} \frac{1}{2\tilde{\mu}} (\Sigma_{ij}^\epsilon - \frac{\tilde{\lambda}}{3\tilde{\lambda} + 2\tilde{\mu}} \Sigma_{kk}^\epsilon \delta_{ij}) \tau_{ij}^\epsilon \\ & + \int_{\Omega^\epsilon} \frac{1}{2\tilde{\mu}} (\Sigma_{ij}^\epsilon - \frac{\lambda^\epsilon}{3\lambda^\epsilon + 2\mu^\epsilon} \Sigma_{kk}^\epsilon \delta_{ij}) \tau_{ij}^\epsilon - \frac{1}{3\tilde{\lambda} + 2\tilde{\mu}} \int_{\mathcal{Q}_\beta^\epsilon} \tilde{\gamma}_1 (\iota_{C^\epsilon}) \tau_{ii}^\epsilon \\ & + \frac{\tilde{\lambda}}{2\tilde{\mu}(3\tilde{\lambda} + 2\tilde{\mu})} \int_{\mathcal{Q}_\beta^\epsilon} (\tilde{\gamma}_2 (\iota_{C^\epsilon}) C_{kk}^\epsilon + \tilde{\gamma}_3 (\iota_{C^\epsilon}) C_{kj}^\epsilon C_{jk}^\epsilon) \tau_{ii}^\epsilon \\ & - \frac{1}{2\tilde{\mu}} \int_{\mathcal{Q}_\beta^\epsilon} (\tilde{\gamma}_2 (\iota_{C^\epsilon}) C_{ij}^\epsilon \tau_{ij}^\epsilon + \tilde{\gamma}_3 (\iota_{C^\epsilon}) C_{ik}^\epsilon C_{kj}^\epsilon \tau_{ij}^\epsilon) \\ & - \frac{1}{3\lambda^\epsilon + 2\mu^\epsilon} \int_{\Omega^\epsilon} \gamma_1^\epsilon (\iota_{C^\epsilon}) \tau_{ii}^\epsilon + \frac{\lambda^\epsilon}{2\mu^\epsilon(3\lambda^\epsilon + 2\mu^\epsilon)} \int_{\Omega^\epsilon} (\gamma_2^\epsilon (\iota_{C^\epsilon}) C_{kk}^\epsilon + \gamma_3^\epsilon (\iota_{C^\epsilon}) C_{kj}^\epsilon C_{jk}^\epsilon) \tau_{ii}^\epsilon \\ & - \frac{1}{2\mu^\epsilon} \int_{\Omega^\epsilon} (\gamma_2^\epsilon (\iota_{C^\epsilon}) C_{ij}^\epsilon \tau_{ij}^\epsilon + \gamma_3^\epsilon (\iota_{C^\epsilon}) C_{ik}^\epsilon C_{kj}^\epsilon \tau_{ij}^\epsilon) = 0, \end{aligned}$$

$$(14) \quad \forall v^\epsilon \in V^\epsilon : \int_{\mathcal{G}^\epsilon} \Sigma_{ij}^\epsilon e_{ij}^\epsilon (v^\epsilon) + \int_{\mathcal{G}^\epsilon} \Sigma_{ij}^\epsilon \partial_i^\epsilon u_k^\epsilon \partial_j^\epsilon v_k^\epsilon - \int_{\mathcal{G}^\epsilon} f_i^\epsilon v_i^\epsilon = 0.$$

Hereafter, we will make use of the constitutive relations in strong form (1)-(2) and of the equilibrium equation in weak form (14). Indeed, all calculations being formal, we are free to choose the most convenient formulation, since they are formally equivalent - see CIMETIERE, GEYMONAT, LE DRET, RAOULT & TUTEK [9] from where this viewpoint is taken for rods.

### 3. Equivalent formulation of the problem $\mathcal{P}^\epsilon$ over two sets independent of $\epsilon$ :

As proposed in CIARLET, LE DRET & NZENGWA [1], we associate with the sets  $\mathcal{Q}$  and  $\Omega^\epsilon$  two disjoint sets, respectively  $\tilde{\Omega}$  and  $\Omega$  as follows : with each point  $x^\epsilon = (x_i^\epsilon) \in \bar{\Omega}^\epsilon$  we associate the point  $x = (x_1, x_2, x_3) = (x_1^\epsilon, x_2^\epsilon, \frac{1}{\epsilon} x_3^\epsilon) \in \bar{\Omega}$  where  $\Omega = \omega \times ]-1, 1[$  ; with each point  $x^\epsilon = (x_i^\epsilon) \in \mathcal{Q}$  we associate the point  $\tilde{x} = (\tilde{x}_i) = (x_i^\epsilon + t) \in \tilde{\Omega}$ , where  $\tilde{\Omega} = \mathcal{Q} + t$ , the vector  $t$  being such that :  $\tilde{\Omega} \cap \bar{\Omega} = \emptyset$ . Let  $\tilde{\omega}_\beta$  be  $\omega_\beta + t$ . Note that two points :  $\tilde{x} \in \tilde{\Omega}_\beta^\epsilon = \Omega_\beta^\epsilon + t$ , and  $x \in \Omega_\beta = \omega_\beta \times ]-1, 1[$ , corresponds to the same point  $x^\epsilon \in \Omega_\beta^\epsilon$ . This correspondence between  $\tilde{x}$  and  $x$  is denoted :  $\tilde{x} \in \tilde{\Omega}_\beta^\epsilon \leftrightarrow x \in \Omega_\beta$ . Other correspondences will be denoted similarly : for example,  $x^\epsilon \in \mathcal{Q} \leftrightarrow \tilde{x} \in \tilde{\Omega}$ .

The following scalings are defined :

$$(15) \quad \begin{cases} u_\alpha^\epsilon(x^\epsilon) = \epsilon^2 u_\alpha(\epsilon)(x) , \\ u_3^\epsilon(x^\epsilon) = \epsilon u_3(\epsilon)(x) , \\ \Sigma_{\alpha\beta}^\epsilon(x^\epsilon) = \epsilon^{-1} \Sigma_{\alpha\beta}(\epsilon)(x) , \text{ for all } x^\epsilon \in \bar{\Omega}^\epsilon \leftrightarrow x \in \bar{\Omega} , \\ \Sigma_{\alpha 3}^\epsilon(x^\epsilon) = \Sigma_{\alpha 3}(\epsilon)(x) , \\ \Sigma_{33}^\epsilon(x^\epsilon) = \epsilon \Sigma_{33}(\epsilon)(x) , \end{cases}$$

$$(16) \quad \begin{cases} u_i^\epsilon(x^\epsilon) = \epsilon \tilde{u}_i(\epsilon)(\tilde{x}), \text{ for all } x^\epsilon \in \mathcal{Q} \leftrightarrow \tilde{x} \in \tilde{\Omega} , \\ \Sigma_{ij}^\epsilon(x^\epsilon) = \epsilon \tilde{\Sigma}_{ij}(\epsilon)(\tilde{x}), \text{ for all } x^\epsilon \in \mathcal{Q}_\beta^\epsilon \leftrightarrow \tilde{x} \in \tilde{\mathcal{Q}}_\beta^\epsilon = (\tilde{\Omega} \setminus \tilde{\Omega}_\beta^\epsilon)^- . \end{cases}$$

Thus,  $u(\epsilon)$ ,  $\tilde{u}(\epsilon)$ ,  $\Sigma_{ij}(\epsilon)$  and  $\chi(\tilde{\mathcal{Q}}_\beta^\epsilon) \tilde{\Sigma}_{ij}(\epsilon)$  are respectively elements of the following sets :

$$V = \{ v \in (W^{1,p}(\Omega))^3 ; v = 0 \text{ on } \Gamma_o = \Gamma_o^1 \} ,$$

$$\tilde{V} = (W^{1,p}(\tilde{\Omega}))^3 ,$$

$$L_s^2 = \{ \tau_{ij} \in (L^2(\Omega))^9 ; \tau_{ij} = \tau_{ji} \} ,$$

$$\tilde{L}_s^2 = \{ \tau_{ij} \in (L^2(\tilde{\Omega}))^9 , \tau_{ij} = \tau_{ji} \} .$$



Moreover, because of the correspondence :  $\tilde{x} \in \tilde{\Omega}_\beta^\epsilon \leftrightarrow x \in \bar{\Omega}_\beta$ , we have the compatibility relations :

$$(17) \quad \begin{aligned} \tilde{u}_\alpha(\epsilon)(\tilde{x}) &= \epsilon u_\alpha(\epsilon)(x), \\ \tilde{u}_3(\epsilon)(\tilde{x}) &= u_3(\epsilon)(x), \end{aligned} \quad \text{for all } \tilde{x} \in \tilde{\Omega}_\beta^\epsilon \leftrightarrow x \in \bar{\Omega}_\beta.$$

Finally, and most importantly, we assume, as in CIARLET, LE DRET & NZENGWA [1], that there exist functions  $f_i \in L^2(\Omega)$  and  $\tilde{f}_i \in L^2(\tilde{\Omega})$  independent of  $\epsilon$ , and functions  $\gamma_i$  defined on  $]0, +\infty[^3$ , independent of  $\epsilon$ , such that :

$$(18) \quad \begin{cases} f_\alpha^\epsilon(x^\epsilon) = \frac{1}{\epsilon} f_\alpha(x), \\ f_3^\epsilon(x^\epsilon) = f_3(x), \\ f_i^\epsilon(x^\epsilon) = \epsilon \tilde{f}_i(\tilde{x}), \end{cases} \quad \begin{aligned} &\text{for all } x^\epsilon \in \Omega^\epsilon \leftrightarrow x \in \Omega, \\ &\text{for all } x^\epsilon \in \quad \leftrightarrow \tilde{x} \in \tilde{\Omega}, \end{aligned}$$

$$(19) \quad \gamma_i^\epsilon = \epsilon^{-3} \gamma_i.$$

Functions  $\tilde{\gamma}_i$  are assumed to be independent of  $\epsilon$ . The reader is referred to CIARLET, LE DRET & NZENGWA [1] for a discussion and mechanical interpretation of these hypotheses.

Using the scalings and assumptions, we obtain new equations in which the sets and the spaces do not depend on  $\epsilon$ .

To formally solve this new problem, we assume now that  $(u(\epsilon), \tilde{u}(\epsilon))$  can be expanded as :

$$(20) \quad (u(\epsilon), \tilde{u}(\epsilon)) = (u, \tilde{u}) + \epsilon (u^1, \tilde{u}^1) + \epsilon^2 (u^2, \tilde{u}^2) + \dots,$$

and so we apply the asymptotic expansion method, successfully used by CIARLET & DESTUYNDER [5], CIARLET [6], DAVET [7] and RAOULT [8] for the case of a single plate. In practice, we will use some of the calculations and results of DAVET [7] and RAOULT [8].

Using (1), (2), (3), (4) and (16) we see that assumption (20) implies the following expansions :

$$(21) \quad \tilde{\Sigma}_{ij}(\epsilon) = \tilde{\Sigma}_{ij} + \epsilon \tilde{\Sigma}_{ij}^1 + \dots,$$

$$(22) \quad \begin{cases} \Sigma_{\alpha\beta}(\epsilon) = \epsilon^{-2} \Sigma_{\alpha\beta}^{-2} + \epsilon^{-1} \Sigma_{\alpha\beta}^{-1} + \Sigma_{\alpha\beta} + \epsilon \Sigma_{\alpha\beta}^1 + \dots, \\ \Sigma_{\alpha 3}(\epsilon) = \epsilon^{-3} \Sigma_{\alpha 3}^{-3} + \epsilon^{-2} \Sigma_{\alpha 3}^{-2} + \epsilon^{-1} \Sigma_{\alpha 3}^{-1} + \Sigma_{\alpha 3} + \epsilon \Sigma_{\alpha 3}^1 + \dots, \\ \Sigma_{33}(\epsilon) = \epsilon^{-4} \Sigma_{33}^{-4} + \epsilon^{-3} \Sigma_{33}^{-3} + \epsilon^{-2} \Sigma_{33}^{-2} + \epsilon^{-1} \Sigma_{33}^{-1} + \Sigma_{33} + \dots \end{cases}$$

In fact, we shall prove that :  $\Sigma_{ij}^{-a} = 0$ , for  $a \in \mathbb{N}^*$ .

For the sake of simplicity, we let :

$$\begin{aligned} t &= \partial_3 u_3, \\ v &= \partial_3 u_3 \left(1 + \frac{1}{2} \partial_3 u_3\right), \\ w &= \partial_3 u_3^1, \\ w^* &= \partial_3 u_3^2. \end{aligned}$$

4. Identification of  $\tilde{\Sigma}_{ij}$ ,  $\Sigma_{ij}^{-a}$  for  $a \in \mathbb{N}^*$ ,  $\Sigma_{\alpha\beta}$ ,  $\tilde{u}$  and  $u$  :  
4.1. Plate part :

Lemma 1 : We have the following expansions :

$$(23) \quad \begin{cases} I_C^\epsilon &= 3 + 2v + 2\epsilon w (1 + \partial_3 u_3) + O(\epsilon^2), \\ II_C^\epsilon &= 3 + 4v + 4\epsilon w (1 + \partial_3 u_3) + O(\epsilon^2), \\ III_C^\epsilon &= 1 + 2v + 2\epsilon w (1 + \partial_3 u_3) + O(\epsilon^2), \text{ on } \Omega, \end{cases}$$

$$(24) \quad \begin{cases} I_C^\epsilon &= 3 + 2\epsilon \partial_i \tilde{u}_i + O(\epsilon^2), \\ II_C^\epsilon &= 3 + 4\epsilon \partial_i \tilde{u}_i + O(\epsilon^2), \\ III_C^\epsilon &= 1 + 2\epsilon \partial_i \tilde{u}_i + O(\epsilon^2), \text{ on } \tilde{\Omega}. \end{cases}$$

Moreover :

$$(25) \quad \begin{cases} \sum_{i=1}^3 \tilde{\gamma}_i (3, 3, 1) = 0, \\ \sum_{i=1}^3 \gamma_i (3, 3, 1) = 0, \end{cases}$$

$$(26) \quad C_{ij}^\epsilon = \delta_{ij} + 2\epsilon \tilde{e}_{ij}(\tilde{u}) + O(\epsilon^2) \text{ in } \tilde{\Omega},$$

$$(27) \quad C_{\alpha\beta}^\epsilon = \delta_{\alpha\beta} + O(\epsilon^2) \text{ in } \Omega,$$

$$(28) \quad C_{\alpha 3}^\epsilon = (2e_{\alpha 3}(u) + \partial_\alpha u_3 \partial_3 u_3) \epsilon + O(\epsilon^2) \text{ in } \Omega,$$

$$(29) \quad C_{33}^\epsilon = 1 + 2v + 2w (1 + \partial_3 u_3) \epsilon + O(\epsilon^2) \text{ in } \Omega,$$

$$(30) \quad C_{\alpha k}^\epsilon C_{k\beta}^\epsilon = \delta_{\alpha\beta} + O(\epsilon^2) \text{ in } \Omega,$$

$$(31) \quad C_{\alpha k}^\epsilon C_{k3}^\epsilon = 2(1 + v) (2e_{\alpha 3}(u) + \partial_\alpha u_3 \partial_3 u_3) \epsilon + O(\epsilon^2) \text{ in } \Omega,$$

$$(32) \quad C_{3k}^\epsilon C_{k3}^\epsilon = (1 + 2v)^2 + 4w (1 + 2v) (1 + \partial_3 u_3) \epsilon + O(\epsilon^2) \text{ in } \Omega.$$

Remark : The notation  $O(\epsilon^2)$  means here that the leading term of the remaining asymptotic expansion is at least of the order  $\epsilon^2$ .

Proof : All these formulas follow from straightforward computations.  $\square$

Now we present the particular form of  $u$ .

Hereafter, we shall assume that :

$$(33) \quad u_3 \in C^1(\bar{\Omega}).$$

Lemma 2 : We have :  $\partial_3 u_3 = 0$  in  $\bar{\Omega}$ , and  $\Sigma_{33}^{-4} = 0$  in  $\Omega$ .

Proof : First of all, we obtain from equation (2) :

$$\epsilon^4 \Sigma_{33}(\epsilon) = \gamma_1 (\iota_C \epsilon) + \gamma_2 (\iota_C \epsilon) C_{33}^\epsilon + \gamma_3 (\iota_C \epsilon) C_{3k}^\epsilon C_{3k}^\epsilon,$$

and, using  $v_3 = \int_{-1}^{x_3} \phi$  where  $\phi \in \mathcal{D}(\Omega)$ , (20), (21) and (22) in (14) :

$$\Sigma_{33}^{-4} (1 + \partial_3 u_3) = 0.$$

Let :

$$F(z) = \sum_{i=1}^3 \gamma_i (3+2z, 3+4z, 1+2z) + 2z [\gamma_2 (3+2z, 3+4z, 1+2z) + 2\gamma_3 (3+2z, 3+4z, 1+2z)] + 4z^2 \gamma_3 (3+2z, 3+4z, 1+2z).$$

Then the function :  $G(t) = (1+t) F[t(1+\frac{t}{2})]$ , satisfies :  $G(t) = 0$ , for all  $t = \partial_3 u_3(x)$  where  $x \in \Omega$ . Now, because of equation (26) :

$$G(0) = F(0) = 0,$$

and :

$$\frac{dG}{dt}(0) = \frac{dF}{dz}(0) + F(0) = \lambda + 2\mu \neq 0.$$

Thus 0 is an isolated zero of  $G$ , and there exists  $\delta > 0$  such that :

$$t(x) = 0 \quad \text{or} \quad |t(x)| \geq \delta \quad \text{for } x \in \Omega.$$

Since  $u_3 \in C^1(\bar{\Omega})$  and  $u_3 = 0$  on  $\Gamma_0$ ,  $t$  is a continuous function and :

$$\partial_3 u_3 = 0 \quad \text{in } \Omega.$$

Then we have :  $\Sigma_{33}^{-4} = 0$  in  $\Omega$ .  $\square$

Lemma 3 : We have :  $\Sigma_{33}^{-2} = \Sigma_{3\alpha}^{-2} = 0$  in  $\Omega$ , and  $\partial_3 u_3^1 = 0$  in  $\Omega$ .

Proof : The term factoring  $\epsilon^{-3}$  in equation (14) yields :

$$2 \int_{\Omega} \Sigma_{\alpha 3}^{-3} e_{\alpha 3}(v) + \int_{\Omega} \Sigma_{33}^{-3} \partial_3 v_3 + \int_{\Omega} \Sigma_{\alpha 3}^{-3} \partial_3 u_3 \partial_3 v_3 = 0.$$

Using firstly  $v_3 = 0$  and  $v_\alpha = \int_{-1}^{x_3} \phi$  where  $\phi \in \mathcal{D}(\Omega)$ , then  $v_\alpha = 0$  and  $v_3 = \int_{-1}^{x_3} \phi$  where  $\phi \in \mathcal{D}(\Omega)$ , we obtain :  $\Sigma_{\alpha 3}^{-3} = \Sigma_{33}^{-3} = 0$ .

Then, equations (2), (26), (30) and (33) imply :

$$\Sigma_{33}^{-3} = w \lambda = 0, \quad \text{and } \lambda \neq 0, \quad \text{thus : } w = 0. \quad \square$$

Lemma 4 : We have :  $\Sigma_{\alpha\beta}^{-2} = \Sigma_{\alpha 3}^{-2} = \Sigma_{33}^{-2} = 0$  in  $\Omega$ , and :  $\partial_{\alpha} u_3 + \partial_3 u_{\alpha} = 0$  in  $\Omega$ .

Proof : First of all, equations (2) and (25) imply that :  $\Sigma_{\alpha\beta}^{-2} = 0$  in  $\Omega$ . Now, using  $v_{\alpha} = \int_{-1}^{x_3} \phi$  where  $\phi \in \mathcal{D}(\Omega)$  and  $v_3 = 0$  in (14), we calculate the term factoring  $\epsilon^{-2}$ , and we obtain :  $\Sigma_{\alpha 3}^{-2} = 0$  in  $\Omega$ . Then, if we set :  $v_{\alpha} = 0$ , and :  $v_3 = \int_{-1}^{x_3} \phi$  where  $\phi \in \mathcal{D}(\Omega)$ , we have :  $\Sigma_{33}^{-2} = 0$  in  $\Omega$ . Finally, it follows from (2) that :

$$\Sigma_{\alpha 3}^{-2} = 2\mu e_{\alpha 3}(u) = 0.$$

□

Due to Lemma 1 and Lemma 3,  $u$  is a Kirchhoff-Love displacement, as has been proved, for example, by CIARLET & RABIER [13] for a single plate ; the equations  $\partial_3 u_3 = 0$  and  $e_{\alpha 3}(u) = 0$  imply that  $\partial_3^2 u_{\alpha} = 0$ , and thus there exist functions  $\zeta_{\alpha}$  and  $\zeta'_{\alpha}$  in the set  $W^{1,p}(\omega)$  such that :

$$u_{\alpha} = \zeta_{\alpha} + x_3 \zeta'_{\alpha} ;$$

as :  $\partial_3 u_{\alpha} = -\partial_{\alpha} u_3 = \zeta'_{\alpha}$ , we see that  $u_3 \in W^{2,p}(\omega)$  and finally :

$$(34) \quad \begin{cases} u_3 = \zeta_3, \\ u_{\alpha} = \zeta_{\alpha} - x_3 \partial_{\alpha} \zeta_3, \end{cases}$$

where  $\zeta_{\alpha} \in W^{1,p}(\omega)$  and  $\zeta_3 \in W^{2,p}(\omega)$ .

Our aim is now to compute the functions  $\Sigma_{\alpha\beta}^{-1}$ ,  $\Sigma_{\alpha\beta}^{-1}$ ,  $\Sigma_{\alpha 3}^{-1}$  and  $\Sigma_{33}^{-1}$ .

Lemma 5 : The following expansions are valid :

$$(35) \quad I_C^{\epsilon} = 3 + 2(\partial_{\alpha} u_{\alpha} + \partial_{\alpha} u_3 \partial_{\alpha} u_3 + w^*) \epsilon^2 + 0(\epsilon^3),$$

$$(36) \quad II_C^{\epsilon} = 3 + 4(\partial_{\alpha} u_{\alpha} + \partial_{\alpha} u_3 \partial_{\alpha} u_3 + w^*) \epsilon^2 + 0(\epsilon^3),$$

$$(37) \quad III_C^{\epsilon} = 1 + 2(\partial_{\alpha} u_{\alpha} + \partial_{\alpha} u_3 \partial_{\alpha} u_3 + w^*) \epsilon^2 + 0(\epsilon^3),$$

$$(38) \quad C_{\alpha\beta}^{\epsilon} = \delta_{\alpha\beta} + (2 e_{\alpha\beta}(u) + \partial_{\alpha} u_3 \partial_{\beta} u_3) \epsilon^2 + 0(\epsilon^3),$$

$$(39) \quad C_{\alpha k}^{\epsilon} C_{k\beta}^{\epsilon} = \delta_{\alpha\beta} + (4 e_{\alpha\beta}(u) + 2 \partial_{\alpha} u_3 \partial_{\beta} u_3) \epsilon^2 + 0(\epsilon^3),$$

$$(40) \quad C_{33}^{\epsilon} = 1 + (\partial_3 u_{\alpha} \partial_3 u_{\alpha} + 2w^*) \epsilon^2 + 0(\epsilon^3),$$

$$(41) \quad C_{3\alpha}^{\epsilon} C_{3\alpha}^{\epsilon} = 0(\epsilon^3).$$

Proof : Using the preceding results, these formulas follow from straightforward computation. □

Lemma 6 : We have the following results :  $\Sigma_{\alpha\beta}^{-1} = \Sigma_{\alpha 3}^{-1} = \Sigma_{33}^{-1} = 0$ , and :

$$(42) \quad \Sigma_{\alpha\beta} = \frac{2\mu\lambda}{\lambda+2\mu} [\partial_{\alpha} u_{\alpha} + \frac{1}{2} \partial_{\alpha} u_3 \partial_{\alpha} u_3] \delta_{\alpha\beta} + 2\mu [e_{\alpha\beta}(u) + \partial_{\alpha} u_3 \partial_{\beta} u_3].$$

Proof : Equation (2) and Lemma 5 imply :

$$\Sigma_{\alpha\beta}(\epsilon) = (\partial_{\alpha} u_{\alpha} + \partial_{\alpha} u_3 \partial_{\alpha} u_3 + w^*) \lambda \delta_{\alpha\beta} + 2(e_{\alpha\beta}(u) + \partial_{\alpha} u_3 \partial_{\beta} u_3)\mu + o(1),$$

and we have :  $\Sigma_{\alpha\beta}^{-1} = 0$ .

Moreover, using  $v_{\alpha} = \int_{-1}^{x_3} \phi$  where  $\phi \in \mathcal{D}(\Omega)$  and  $v_3 = 0$ , then  $v_3 = \int_{-1}^{x_3} \phi$  and  $v_{\alpha} = 0$  in the term factoring  $\epsilon^{-1}$  in equation (14), we obtain :

$$\Sigma_{\alpha 3}^{-1} = \Sigma_{33}^{-1} = 0.$$

Since equation (2) implies :

$$\Sigma_{33}^{-2} = 0 = (\partial_{\alpha} u_{\alpha} + \partial_{\alpha} u_3 \partial_{\alpha} u_3 + w^*) \lambda + \mu (\partial_3 u_{\alpha} \partial_3 u_{\alpha} + 2w^*),$$

$$w^* = -\frac{1}{\lambda + 2\mu} [\lambda(\partial_{\alpha} u_{\alpha} + \partial_{\alpha} u_3 \partial_{\alpha} u_3) + \mu \partial_{\alpha} u_3 \partial_{\alpha} u_3].$$

Consequently, we obtain the result (42). □

We have proved that, under the assumption (20),  $(\tilde{\Sigma}(\epsilon), \Sigma(\epsilon), \tilde{u}(\epsilon), u(\epsilon))$  admits the following expansion :

$$(\tilde{\Sigma}(\epsilon), \Sigma(\epsilon), \tilde{u}(\epsilon), u(\epsilon)) = (\tilde{\Sigma}, \Sigma, \tilde{u}, u) + \epsilon (\tilde{\Sigma}^1, \Sigma^1, \tilde{u}^1, u^1) + \dots$$

#### 4.2. Three-dimensional part :

Lemma 7 : We have :

$$(43) \quad \tilde{\Sigma}_{ij} = 2 \tilde{\mu} \tilde{e}_{ij}(\tilde{u}) + \delta_{ij} \tilde{\lambda} \tilde{e}_{pp}(\tilde{u}) \text{ in } \tilde{\Omega} \setminus (\omega_{\beta} + t) = \tilde{\Omega}_{\beta}.$$

Proof : From equation (2), formulas (16) and (25) imply :

$$\epsilon \tilde{\Sigma}_{ij}(\epsilon) = \tilde{e}_{pp}(\tilde{u}) \delta_{ij} \left[ \sum_{i=1}^3 2 \tilde{\partial}_1 \tilde{\gamma}_i(3,3,1) + 4 \tilde{\partial}_2 \tilde{\gamma}_1(3,3,1) + 2 \tilde{\partial}_3 \tilde{\gamma}_1(3,3,1) \right] + 2 \tilde{e}_{ij}(\tilde{u}) [\tilde{\gamma}_2(3,3,1) + 2 \tilde{\gamma}_3(3,3,1)] + o(\epsilon^2).$$

Therefore, we obtain (43). □

#### 4.3. Junction conditions satisfied by $(u, \tilde{u})$ :

The preceding information about  $\tilde{\Sigma}_{ij}$  and  $\Sigma_{\alpha\beta}$  have been obtained from (1) and (2), which are the constitutive laws of the materials. We now proceed to explore the consequences of the equilibrium equation (14). As proposed by C. APLET, LE DRET & NZENGWA [1], we define :

$$V(\epsilon) = \{(\tilde{v}(\epsilon), v(\epsilon)) \in \tilde{V} \times V ; \tilde{v}_{\alpha}(\epsilon)(\tilde{x}) = \epsilon v_{\alpha}(\epsilon)(x), \tilde{v}_3(\epsilon)(\tilde{x}) = v_3(\epsilon)(x), \\ \text{for all } \tilde{x} \in \tilde{\Omega}_{\beta}^{\epsilon} \leftrightarrow x \in \Omega_{\beta}\}.$$

We see that  $V(\epsilon)$  is a subspace of  $\tilde{V} \times V$  whose elements satisfy the transmission conditions (17).

Equilibrium equation (14) becomes :

$$(44) \quad \forall (\tilde{v}(\epsilon), v(\epsilon)) \in V(\epsilon) :$$

$$\begin{aligned} & \int_{\tilde{\Omega}} \chi(\tilde{\mathcal{Q}}_{\beta}^{\epsilon}) \tilde{\Sigma}_{ij}(\epsilon) \tilde{e}_{ij}(\tilde{v}(\epsilon)) + \epsilon \int_{\tilde{\Omega}} \chi(\tilde{\mathcal{Q}}_{\beta}^{\epsilon}) \tilde{\Sigma}_{ij}(\epsilon) \tilde{\partial}_i \tilde{u}_{\ell}(\epsilon) \tilde{\partial}_j \tilde{v}_{\ell}(\epsilon) \\ & - \int_{\tilde{\Omega}} \chi(\tilde{\mathcal{Q}}_{\beta}^{\epsilon}) \tilde{f}_i \cdot \tilde{v}_i(\epsilon) + \int_{\Omega} \Sigma_{ij}(\epsilon) e_{ij}(v(\epsilon)) + \int_{\Omega} \Sigma_{ij}(\epsilon) \partial_i u_3(\epsilon) \partial_j v_3(\epsilon) \\ & + \epsilon^2 \int_{\Omega} \Sigma_{ij}(\epsilon) \partial_i u_{\gamma}(\epsilon) \partial_j v_{\gamma}(\epsilon) - \int_{\Omega} f_i \cdot v_i(\epsilon) = 0. \end{aligned}$$

The following lemma gives the junction condition satisfied by  $(\tilde{u}, u)$ .

Lemma 8 : The pair  $(\tilde{u}, u)$  satisfies :

$$(45) \quad \text{tr } \tilde{u}_{\alpha} |_{\tilde{\omega}_{\beta}} = 0,$$

$$(46) \quad \text{tr } \tilde{u}_3 |_{\tilde{\omega}_{\beta}} = \text{tr } u_3 |_{\omega_{\beta}} = \zeta_3 |_{\omega_{\beta}}.$$

Proof : By definition of the space  $V(\epsilon)$ , we have :

$$\tilde{u}_{\alpha}(\epsilon) |_{\tilde{\omega}_{\beta}} = \epsilon u_{\alpha}(\epsilon) |_{\omega_{\beta}}, \text{ and } \tilde{u}_3(\epsilon) |_{\tilde{\omega}_{\beta}} = u_3(\epsilon) |_{\omega_{\beta}}.$$

The asymptotic expansion (20) yields the result. □

#### 4.4. Identification of the limit problem :

Our aim now is to identify a limit problem whose unknowns should be  $\tilde{\Sigma}$ ,  $\Sigma$ ,  $\tilde{u}$  and  $u$ . According to (34), let :

$$\begin{aligned} V_{KL}(\Omega) &= \{v \in W^{1,p}(\Omega), v = 0 \text{ on } \Gamma_o, \text{ such that there exist } \eta_{\alpha} \in W^{1,p}(\omega) \\ & \text{ and } \eta_3 \in W^{2,p}(\omega) \text{ such that :} \\ & v_{\alpha} = \eta_{\alpha} - x_3 \partial_{\alpha} \eta_3, \\ & v_3 = \eta_3, \\ & \eta_1 = \partial_{\nu} \eta_3 = 0 \text{ on } \gamma_o.\} \end{aligned}$$

$$\begin{aligned} \tilde{V}_{\beta} \times V_{KL}(\Omega) &= \{(\tilde{v}, v) \in \tilde{V} \times V_{KL}(\Omega) ; \text{tr } \tilde{v}_{\alpha} |_{\tilde{\omega}_{\beta}} = 0, \text{tr } \tilde{v}_3 |_{\tilde{\omega}_{\beta}} = \eta_3 \text{ where} \\ & \eta_1 \text{ are associated to } v \text{ by definition of : } v \in V_{KL}(\Omega).\}. \end{aligned}$$

Hereafter,  $\tilde{\omega}$  denotes the intersection of the set  $\tilde{\Omega}$  by the plane that contains  $\tilde{\omega}_{\beta}$ .

Lemma 9 : Let  $(\tilde{v}, v) \in \tilde{V}^{\beta} \times V_{KL}(\Omega)$  be such that, either  $\text{supp } \tilde{v}$  is contained in the set :  $\{ \tilde{x} = (\tilde{x}_1) \in \tilde{\Omega} ; \tilde{x}_1 \leq 0 \}$  and  $v = 0$ , or  $\tilde{v} |_{\tilde{\omega}} \in W^{1,p}(\tilde{\omega})$ . Then there exists a sequence  $(\tilde{v}(\epsilon), v(\epsilon))$  such that :

$(\tilde{v}(\epsilon), v(\epsilon)) \in V(\epsilon)$  for all  $\epsilon > 0$ ,

$v(\epsilon) \in V_{KL}(\Omega)$  for all  $\epsilon > 0$ ,

$$\|v(\epsilon) - v\|_{W^{1,p}(\Omega)} \xrightarrow{\epsilon \rightarrow 0} 0,$$

$$\|\tilde{v}(\epsilon) - \tilde{v}\|_{W^{1,p}(\tilde{\Omega})} \xrightarrow{\epsilon \rightarrow 0} 0.$$

Proof : It is the same as in CIARLET, LE DRET, NZENGWA [1] since the results used from NEČAS [14] p.80 extend to the present situation -  $p \geq 4$ .  $\square$

Lemma 10 : Let  $(\tilde{v}, v) \in \tilde{V}^\beta \times V_{KL}(\Omega)$  be such that, either  $\text{supp } \tilde{v}$  is contained in the set  $\{\tilde{x} = (\tilde{x}_1) \in \tilde{\Omega} ; \tilde{x}_1 \leq 0\}$  and  $v = 0$ , or  $\tilde{v}|_{\tilde{\omega}} \in W^{1,p}(\tilde{\omega})$ . Then the couple  $(\tilde{u}, u)$  satisfies :

$$(47) \quad \int_{\tilde{\Omega}} \tilde{\Sigma}_{ij} \tilde{e}_{ij}(\tilde{v}) - \int_{\tilde{\Omega}} \tilde{f}_i \cdot \tilde{v}_i + \int_{\omega} n_{\alpha\beta} \partial_{\alpha} \zeta_3 \partial_{\beta} \eta_3 + \int_{\omega} n_{\alpha\beta} \left( \frac{\partial_{\alpha} \eta_{\beta} + \partial_{\beta} \eta_{\alpha}}{2} \right) \\ - \int_{\omega} m_{\alpha\beta} \partial_{\alpha\beta} \eta_3 - \int_{\omega} f_i^0 \eta_i + \int_{\omega} f_{\alpha}^1 \partial_{\alpha} \eta_3 = 0,$$

where :  $v_{\alpha} = \eta_{\alpha} - x_3 \partial_{\alpha} \eta_3$  ,  $v_3 = \eta_3$  ,

$u$  is given by (34) ,

$\tilde{\Sigma}_{ij}$  is given by (43) ,

and where :

$$(48) \quad m_{\alpha\beta} = \int_{-1}^1 x_3 \Sigma_{\alpha\beta} = - \frac{4\mu}{3} \left( \partial_{\alpha\beta} \zeta_3 + \frac{\lambda}{\lambda+2\mu} \Delta \zeta_3 \delta_{\alpha\beta} \right) ,$$

$$(49) \quad n_{\alpha\beta} = \int_{-1}^1 \Sigma_{\alpha\beta} = 4 \mu \left( \left( \frac{\partial_{\alpha} \delta_{\beta} + \partial_{\beta} \delta_{\alpha}}{2} \right) + \frac{\lambda}{\lambda+2\mu} \partial_{\mu} \zeta_3 \delta_{\alpha\beta} \right) \\ + 2 \mu \left( \partial_{\alpha} \zeta_3 \partial_{\beta} \zeta_3 + \frac{\lambda}{\lambda+2\mu} \partial_{\mu} \zeta_3 \partial_{\mu} \zeta_3 \delta_{\alpha\beta} \right) ,$$

$$(50) \quad f_i^0(x_1, x_2) = \int_{-1}^1 f_i(x_1, x_2, s) ds,$$

$$(51) \quad f_i^1(x_1, x_2) = \int_{-1}^1 f_i(x_1, x_2, s) s ds.$$

Proof : We use the sequence  $(\tilde{v}(\epsilon), v(\epsilon))$  constructed in Lemma 9 for approximating the couple  $(\tilde{v}, v)$  as test functions in the variational equation (44). This equation reduces to :

$$\int_{\tilde{\Omega}} \chi(\tilde{\mathcal{O}}_{\beta}^{\epsilon}) \tilde{\Sigma}_{ij} \tilde{e}_{ij}(\tilde{v}(\epsilon)) - \int_{\tilde{\Omega}} \chi(\tilde{\mathcal{O}}_{\beta}^{\epsilon}) \tilde{f}_i \cdot \tilde{v}_i(\epsilon) \\ + \int_{\Omega} \Sigma_{\alpha\beta} e_{\alpha\beta}(v(\epsilon)) + \int_{\Omega} \Sigma_{\alpha\beta} \partial_{\alpha} u_3 \partial_{\beta} v_3(\epsilon) - \int_{\Omega} f_i \cdot v_i(\epsilon) + o(\epsilon) = 0.$$

Then, when  $\epsilon$  tends to zero, we obtain (47).  $\square$

Let :  $[W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_{\beta} = \{(\tilde{v}, \eta_3) \in W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega) ; \eta_3|_{\gamma_0} = \partial_{\nu} \eta_3 = 0, \text{tr } \tilde{v}|_{\tilde{\omega}} = \eta_3|_{\omega}, \text{tr } \tilde{v}|_{\tilde{\omega}} = 0\}.$

Lemma 11 : For any  $(\tilde{v}, \eta_3)$  that belongs to the space  $[W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_{\beta}$ , we have :

$$(52) \quad \int_{\tilde{\Omega}} \tilde{\Sigma}_{ij} \tilde{e}_{ij}(\tilde{v}) - \int_{\tilde{\Omega}} \tilde{f}_i \cdot \tilde{v}_i + \int_{\omega} n_{\alpha\beta} \partial_{\alpha} \zeta_3 \partial_{\beta} \eta_3 - \int_{\omega} m_{\alpha\beta} \partial_{\alpha\beta} \eta_3 - \int_{\omega} f_3^0 \eta_3 + \int_{\omega} f_{\alpha}^1 \partial_{\alpha} \eta_3 = 0.$$

Proof : The same idea as in CIARLET, LE DRET & NZENGWA [1] is used.

Let  $\tilde{\eta}_3 \in W^{2,p}(\tilde{\omega})$  denote an extension of  $\eta_3|_{\tilde{\omega}}$  and let :

$$\tilde{w}_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \tilde{\eta}_3(\tilde{x}_1, \tilde{x}_2) \text{ in } \tilde{\Omega}.$$

Then  $(\tilde{w}^*, \eta_3) = ((0, 0, \tilde{w}_3), \eta_3)$  satisfies the conditions of Lemma 10, and thus satisfies the equation (52). Though the first equations used were non linear, we observe that equation (52) is linear with respect to  $(\tilde{v}, \eta_3)$ , which is why the proof of the authors mentioned above can be extended here.

Let us recall that, in order to prove that equation (52) is satisfied for all pair  $(\tilde{v}, 0) \in [W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_{\beta}$ , two sequences  $\tilde{r}_n$  and  $\tilde{s}_n$  are used :

$$\begin{aligned} \tilde{r}_n &\in W^{1,p}(\tilde{\Omega}), \tilde{r}_n|_{\tilde{\omega}} \in W^{1,p}(\tilde{\omega}), \\ \tilde{s}_n &\in W^{1,p}(\tilde{\Omega}), \text{supp } \tilde{s}_n \subset \{\tilde{x} = (\tilde{x}_1) \in \tilde{\Omega} ; \tilde{x}_1 \leq 0\}, \\ \tilde{r}_n + \tilde{s}_n &\xrightarrow{n \rightarrow +\infty} \tilde{v} \text{ in } W^{1,p}(\tilde{\Omega}). \end{aligned}$$

The equation (52) is separately satisfied by  $((\tilde{r}_n)_n, 0)$  and  $((\tilde{s}_n)_n, 0)$ , and since it is linear and continuous with respect to  $\tilde{v} \in H^1(\tilde{\Omega})$ , the conclusion follows.

The constructions of  $\tilde{r}_n$  and  $\tilde{s}_n$  are given in CIARLET, LE DRET & NZENGWA [1], as is the proof of the convergence  $\tilde{r}_n + \tilde{s}_n \xrightarrow{n \rightarrow +\infty} \tilde{v}$  in  $W^{1,p}(\tilde{\Omega})$ , since the continuity of the translation operator used in  $L^2(\tilde{\Omega})$  for that proof is also verified in  $L^p(\tilde{\Omega})$  - see J. NEČAS [14] p.57.  $\square$

Lemma 12 : For all  $(\eta_1, \eta_2) \in W(\omega) = \{(\eta_1, \eta_2) \in (W^{1,p}(\omega))^2 ; \eta_{\alpha} = 0 \text{ on } \gamma_0\}$ , we have :

$$(53) \quad \frac{1}{2} \int_{\omega} n_{\alpha\beta} (\zeta_{\alpha} \eta_{\beta} + \zeta_{\beta} \eta_{\alpha}) - \int_{\omega} f_{\alpha}^0 \eta_{\alpha} = 0$$

Proof : This is a simple consequence of (47).  $\square$

Formally, we have identified  $\tilde{\Sigma}_{ij}$ ,  $\Sigma_{\alpha\beta}$ ,  $\tilde{u}$  and  $u$ , given by the following equations. From equation (52) is deduced :



$$(54) \quad \begin{cases} - \tilde{\partial}_j \tilde{\Sigma}_{ij} = \tilde{f}_i \text{ in } \tilde{\Omega}, \\ \tilde{\Sigma}_{ij} \cdot \tilde{n}_j = 0 \text{ on } \partial\tilde{\Omega} \setminus \partial\tilde{\omega}_\beta, \end{cases}$$

and  $\tilde{\Sigma}_{ij}$  is given by (43) ;

$$(55) \quad \begin{cases} - \partial_{\alpha\beta} m_{\alpha\beta} - \partial_\beta (n_{\alpha\beta} \partial_\alpha \zeta_3) = f_3^0 + \partial_\alpha f_\alpha^1 - \chi(\omega_\beta) (\tilde{\Sigma}_{33}^+ \tilde{n}_3|_{\omega_\beta^+} + \tilde{\Sigma}_{33}^- \tilde{n}_3|_{\omega_\beta^-}) \\ \text{in } \omega, \\ m_{\alpha\beta} \nu_\alpha \nu_\beta = 0, \quad n_{\alpha\beta} \nu_\alpha \tau_\beta = 0 \text{ on } \gamma_0, \\ - (\partial_\alpha m_{\alpha\beta} + n_{\alpha\beta} \partial_\alpha \zeta_3) \nu_\beta = f_\alpha^1 \nu_\alpha \text{ on } \gamma_0, \\ \zeta_3 = \partial_\nu \zeta_3 = 0 \text{ on } \gamma_0, \end{cases}$$

where  $\omega_\beta^+$  and  $\omega_\beta^-$  denote respectively the "upper" and "lower" faces of the set  $\omega_\beta$  ; the outer normal vectors of  $\partial\tilde{\Omega} \setminus \partial\tilde{\omega}_\beta$  along  $\omega_\beta^+$  and  $\omega_\beta^-$  are denoted by  $\tilde{n}_3|_{\omega_\beta^+}$  and  $\tilde{n}_3|_{\omega_\beta^-}$  respectively. The expressions of  $m_{\alpha\beta}$  and  $n_{\alpha\beta}$  are given by (48) and (49).

Then, from (53), we obtain :

$$(56) \quad \begin{cases} - \partial_\alpha n_{\alpha\beta} = f_\beta^0 \text{ in } \omega, \\ n_{\alpha\beta} \nu_\beta = 0 \text{ on } \gamma_1 = \partial\omega \setminus \gamma_0, \\ \delta_\alpha = 0 \text{ on } \gamma_0. \end{cases}$$

Remark : As opposed to the linear case, the limit problem does not split into two separate problems, one in the unknowns  $(\tilde{u}, \zeta_3)$  and the other in the unknowns  $(\zeta_1, \zeta_2)$ , but it remains coupled through nonlinear terms.

We can now wonder whether these equations admit a solution.

Lemma 13 : The equations (52), (53) admit at least one solution in the vector space  $[W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_\beta \times W(\omega)$ .

Proof : The arguments are taken from CIARLET & RABIER [13], Theorem 2.1.1, in which the reader will find all the details of the following proof. Let us denote  $(\eta_1, \eta_2)$  by  $\eta_0$ .

Using (41), (48) and (49) in (52) and (53), the problem to solve take the form :

$$(57) \quad \begin{cases} \frac{\partial J((\tilde{u}, \zeta_3), \zeta_0)}{\partial (\tilde{u}, \zeta_3)} (\tilde{v}, \eta_3) = 0, \quad \forall (\tilde{v}, \eta_3) \in [W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_\beta, \\ \frac{\partial J((\tilde{u}, \zeta_3), \zeta_0)}{\partial \zeta_0} (\eta_0) = 0, \quad \forall \eta_0 \in W(\omega), \end{cases}$$

with :

$$(58) \quad J((\tilde{v}, \eta_3), \eta_0) = \frac{1}{2} \int_{\tilde{\Omega}} (\tilde{\lambda} \tilde{e}_{pp}(\tilde{v}) \tilde{e}_{qq}(\tilde{v}) + 2 \tilde{\mu} \tilde{e}_{ij}(\tilde{v}) \tilde{e}_{ij}(\tilde{v})) \\ + \frac{2}{3} \mu \int_{\omega} \left( \frac{\lambda}{\lambda+2\mu} (\Delta \eta_3)^2 + 2 (\partial_{12} \eta_3)^2 \right) \\ + 2 \mu \int_{\omega} \left( \sum_{\alpha, \beta} (\gamma_{\alpha\beta}(\eta_0) + \frac{\partial_{\alpha} \eta_3 \partial_{\beta} \eta_3}{2})^2 + \frac{\lambda}{\lambda+2\mu} (\partial_{\mu} \eta_{\mu} + \frac{\partial_{\alpha} \eta_3 \partial_{\beta} \eta_3}{2})^2 \right) \\ + \mu \int_{\omega} \left( \sum_{\alpha, \beta} (\gamma_{\alpha\beta}(\eta_0))^2 + \frac{\lambda}{\lambda+2\mu} (\partial_{\mu} \eta_{\mu})^2 \right) \\ - \int_{\tilde{\Omega}} \tilde{f}_i \cdot \tilde{v}_i - \int_{\omega} f_{\alpha}^0 \cdot \eta_{\alpha} - \int_{\omega} f_3^0 \eta_3 + \int_{\omega} f_{\alpha}^1 \partial_{\alpha} \eta_3.$$

$J$  is a weakly lower semi-continuous functional, since some terms define convex continuous, therefore weakly l.s.c. functionals in  $[W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_{\beta} \times W(\omega)$ , and the other terms are weakly continuous because  $W^{2,p}(\omega)$  is compactly imbedded in  $W^{1,p}(\omega)$ . Thus, we shall prove that :

$$(59) \quad J((\tilde{v}, \eta_3), \eta_0) \leq 0 \Rightarrow \|(\tilde{v}, \eta_3)\|_{\beta} \leq K,$$

where  $K$  is a positive constant and where the mapping :

$$\| \cdot \|_{\beta} : (\tilde{v}, \eta_3) \rightarrow (|\tilde{e}_{ij}(\tilde{v})|_{0, \tilde{\Omega}}^2 + \sum_{\alpha, \beta} |\partial_{\alpha\beta} \eta_3|_{0, \omega}^2)^{1/2},$$

is a norm over  $[W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_{\beta}$  equivalent to the norm :

$$(\tilde{v}, \eta_3) \rightarrow (\|\tilde{v}\|_{W^{1,p}(\tilde{\Omega})}^2 + \|\eta_3\|_{W^{2,p}(\omega)}^2)^{1/2}.$$

By contradiction, let us assume that there exists a sequence  $((\tilde{v}^n, \eta_3^n), \eta_0^n)$  such that :

$$J((\tilde{v}^n, \eta_3^n), \eta_0^n) \leq 0,$$

$$\lim_{n \rightarrow +\infty} \|\eta_3^n\|_{W^{2,p}(\omega)} = +\infty.$$

Adapting the computations of CIARLET & RABIER [13], we can obtain :

$$\|(\tilde{v}^n, \eta_3^n)\|_{\beta} \leq M p(\eta_3^n),$$

where  $M$  is a positive constant, and :  $p(\eta_3) = [\int_{\omega} (\partial_{\alpha} \eta_3 \partial_{\beta} \eta_3 \partial_{\alpha} \eta_3 \partial_{\beta} \eta_3)]^{1/4}$ .

The set :  $T_M = \{(\tilde{v}, \eta_3) \in [W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_{\beta}, \|(\tilde{v}, \eta_3)\|_{\beta} \leq M p(\eta_3)\}$  is a weakly closed cone in which each sequence that converges weakly to a couple  $(\tilde{v}, 0)$  converges strongly towards  $(0, 0)$ . Using this cone, it can be established that, for any  $\epsilon > 0$  and  $M > 0$ , there exists a positive constant  $C = C(\epsilon, M)$  such that :

$$\sum_{\alpha, \beta} |e_{\alpha\beta}(\eta_0) + \frac{1}{2} \partial_{\alpha} \eta_3 \partial_{\beta} \eta_3|_{0, \omega}^2 \geq C (\|\eta_0\|_{W(\omega)}^2 + \|(\tilde{v}, \eta_3)\|_{\beta}^2),$$

for all  $((\tilde{v}, \eta_3), \eta_0) \in [W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_{\beta} \times W(\omega)$  such that  $(\tilde{v}, \eta_3) \in T_M$  and  $\|\eta_0\|_{W(\omega)}^2 + \|(\tilde{v}, \eta_3)\|_{\beta}^2 \geq \epsilon^2$  - see CIARLET & RABIER [13], lemma 2.1.2.

The contradiction follows, and so the conclusion that :

$B = \{(\tilde{v}, \eta_3), \eta_0\} ; J((\tilde{v}, \eta_3), \eta_0) \leq 0\}$  is bounded.

Therefore, it exists  $((\tilde{u}, \zeta_3), \zeta_0) \in [W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_{\beta} \times W(\omega)$  such that :

$$J((\tilde{u}, \zeta_3), \zeta_0) \leq J((\tilde{v}, \eta_3), \eta_0),$$

for all  $((\tilde{v}, \eta_3), \eta_0)$  in the set  $[W^{1,p}(\tilde{\Omega}) \times W^{2,p}(\omega)]_{\beta} \times W(\omega)$ , and this gives the conclusion of the proposition.  $\square$

##### 5. Identification of $\Sigma_{\alpha 3}$ and $\Sigma_{33}$ :

We are now interested by the computation of the functions  $\Sigma_{\alpha 3}$  and  $\Sigma_{33}$ . Firstly, let us examine the equation (11) to establish some transmission conditions along  $\tilde{\omega}_{\beta}$ . Using the scalings (15) and (16), we obtain :

$$\begin{aligned} \text{for } i = 1, 2 : \epsilon \tilde{\Sigma}_{\alpha 3}(\epsilon) + \epsilon^2 \tilde{\Sigma}_{k3}(\epsilon) \tilde{\partial}_k \tilde{u}_{\alpha}(\epsilon) &= \Sigma_{\alpha 3}(\epsilon) + \epsilon^2 \Sigma_{\beta 3}(\epsilon) \partial_{\beta} u_{\alpha}(\epsilon) \\ &+ \epsilon^2 \Sigma_{33}(\epsilon) \partial_{\beta} u_3(\epsilon), \end{aligned}$$

$$\begin{aligned} \text{for } i = 3 : \epsilon \tilde{\Sigma}_{33}(\epsilon) + \epsilon^2 \tilde{\Sigma}_{k3}(\epsilon) \tilde{\partial}_k \tilde{u}_3(\epsilon) &= \epsilon \Sigma_{33}(\epsilon) + \epsilon \Sigma_{\alpha 3}(\epsilon) \partial_{\alpha} u_3(\epsilon) \\ &+ \epsilon \Sigma_{33}(\epsilon) \partial_3 u_3(\epsilon). \end{aligned}$$

Using the results of part 4, we have :

$$(60) \quad \Sigma_{\alpha 3} = 0 \quad \text{on } \Gamma_{\beta}^{+} \quad \text{where } \Gamma_{\beta}^{+} = \omega_{\beta} \times \{+1\},$$

$$(61) \quad \tilde{\Sigma}_{33} = \Sigma_{33} \quad \text{on } \Gamma_{\beta}^{+},$$

$$\text{and also : } \tilde{\Sigma}_{\alpha 3} = \Sigma_{\alpha 3}^1 \quad \text{on } \Gamma_{\beta}^{+}.$$

In the equations (44), let us choose a sequence  $(\tilde{v}(\epsilon), v(\epsilon)) \in V(\epsilon)$  of test-functions, for all  $\epsilon > 0$ , constructed from  $(\tilde{v}, v) = (0, (v_1, v_2, 0))$  where  $v_{\alpha} \in W_0^{1,p}(\Omega)$ , as following :

$$\left\{ \begin{array}{l} \tilde{v}_3(\epsilon) = 0 \text{ in } \tilde{\Omega}, \\ \tilde{v}_{\alpha}(\epsilon)(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \epsilon v_{\alpha}(x_1, x_2, x_3) \text{ for all } \tilde{x} \in \tilde{\Omega}_{\beta}^{\epsilon} \leftrightarrow x \in \Omega_{\beta}, \\ v_{\alpha}(\epsilon) = 0 \text{ in } \tilde{\Omega} \setminus \tilde{\Omega}^{\epsilon}, \\ v(\epsilon) = v \text{ in } \Omega. \end{array} \right.$$

We can easily see that  $(\tilde{v}(\epsilon), v(\epsilon)) \in V^{\epsilon}$  for all  $\epsilon > 0$ , and that :

$$\| \tilde{v}(\epsilon) \|_{W^{1,p}(\tilde{\Omega})} \xrightarrow{\epsilon \rightarrow 0} 0.$$

Thus, (44) is reduced to :

$$\int_{\Omega} \Sigma_{\alpha\beta} e_{\alpha\beta}(v) + \int_{\Omega} \Sigma_{\alpha 3} \partial_3 v_{\alpha} - \int_{\Omega} f_{\alpha} \cdot v_{\alpha} = 0.$$

Then again, equations (48) and (49) give :

$$(62) \quad \Sigma_{\alpha\beta} = \frac{1}{2} n_{\alpha\beta} + \frac{3}{2} x_3 m_{\alpha\beta}.$$

Using (62) in the preceding equation, and denoting, for the sake of simplicity :  $\int_{-1}^1 f = \int_{-1}^1 f(x_1, x_2, s) ds$  for any function  $f$  defined on  $\Omega$ , we obtain :

$$\frac{1}{2} \int_{\omega} n_{\alpha\beta} \left( \int_{-1}^1 \partial_{\alpha} v_{\beta} \right) + \frac{3}{2} \int_{\omega} m_{\alpha\beta} \left( \int_{-1}^1 s \partial_{\alpha} v_{\beta} \right) + \int_{\Omega} \Sigma_{\alpha 3} \partial_3 v_{\alpha} - \int_{\Omega} f_{\alpha} v_{\alpha} = 0.$$

Because of (56), we can write :

$$\int_{\omega} n_{\alpha\beta} \left( \int_{-1}^1 \partial_{\alpha} v_{\beta} \right) = \int_{\omega} f_{\beta}^0 \left( \int_{-1}^1 v_{\beta} \right), \text{ for all } v_{\alpha} \in W_0^{1,p}(\Omega).$$

Finally, we have :

$$- \int_{\Omega} \partial_3 \Sigma_{\alpha 3} \cdot v_{\alpha} + \int_{\Omega} \left( \frac{1}{2} f_{\alpha}^0 - f_{\alpha} \right) - \frac{3}{2} \int_{\Omega} x_3 \partial_{\beta} m_{\alpha\beta} \cdot v_{\alpha} = 0, \text{ for all } v_{\alpha} \in W_0^{1,p}(\Omega).$$

Then, using (60) and (10) :

$$(63) \quad \Sigma_{\alpha 3} = \frac{3}{4} (1-x_3^2) \partial_{\beta} m_{\alpha\beta} + \frac{1}{2} f_{\alpha}^0 (x_3 + 1) - \int_{-1}^{x_3} f_{\alpha}.$$

Now, we do the same thing with  $\tilde{v} = 0$  and  $v = (0, 0, v_3)$ , with  $v_3 \in W_0^{2,p}(\Omega)$ . The sequence  $(\tilde{v}(\epsilon), v(\epsilon))$  is constructed as follows :

$$\begin{cases} \tilde{v}_{\alpha}(\epsilon) = 0 \text{ in } \tilde{\Omega}, \\ \tilde{v}_3(\epsilon)(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = v_3(x_1, x_2, x_3) \text{ for all } \tilde{x} \in \tilde{\Omega}_{\beta}^{\epsilon} \leftrightarrow x \in \Omega_{\beta}, \\ \tilde{v}_3(\epsilon) = 0 \text{ in } \tilde{\Omega} \setminus \tilde{\Omega}_{\beta}^{\epsilon}, \\ v(\epsilon) = v \text{ in } \Omega. \end{cases}$$

The pair  $(\tilde{v}(\epsilon), v(\epsilon)) \in V^{\epsilon}$  for all  $\epsilon > 0$ . Thus, (45) is reduced to :

$$\int_{\Omega} \Sigma_{\alpha 3} \partial_{\alpha} v_3 + \int_{\Omega} \Sigma_{33} \partial_3 v_3 + \int_{\Omega} \Sigma_{\alpha j} \partial_{\alpha} \zeta_3 \partial_j v_3 - \int_{\Omega} f_3 v_3 = 0,$$

for all  $v_3 \in W^{2,p}(\Omega)$  with  $v_3 = 0$  on  $\Gamma_0$ .

$$\begin{aligned} \text{Therefore : } \partial_3 \Sigma_{33} &= - \partial_{\alpha} \Sigma_{\alpha 3} - \partial_j (\Sigma_{\alpha j} \partial_{\alpha} \zeta_3) - f_3 \\ &= - \frac{3}{4} (1-x_3^2) \partial_{\alpha\beta} m_{\alpha\beta} - \frac{3}{2} x_3 m_{\alpha\beta} \partial_{\alpha\beta} \zeta_3 - \partial_{\beta} \left( \frac{1}{2} \partial_{\alpha} \zeta_3 n_{\alpha\beta} \right) \\ &\quad - \frac{1}{2} \partial_{\alpha} f_{\alpha}^0 (x_3+1) - \frac{1}{2} \partial_{\alpha} \zeta_3 f_{\alpha}^0 + \partial_{\alpha} \zeta_3 f_{\alpha} + \int_{-1}^{x_3} \partial_{\alpha} f_{\alpha} - f_3, \end{aligned}$$

by using (60), (62) and (63).

This equation can be integrated, and we obtain :

$$\begin{aligned} \Sigma_{33} &= - \frac{3}{4} (x_3 - \frac{1}{3} x_3^3) \partial_{\alpha\beta} m_{\alpha\beta} - \frac{3}{4} x_3^2 m_{\alpha\beta} \partial_{\alpha\beta} \zeta_3 - x_3 \partial_{\beta} \left( \frac{1}{2} \partial_{\alpha} \zeta_3 n_{\alpha\beta} \right) \\ &\quad - \frac{1}{2} \partial_{\alpha} f_{\alpha}^0 \left( \frac{1}{2} x_3^2 + x_3 \right) - \frac{1}{2} x_3 \partial_{\alpha} \zeta_3 f_{\alpha}^0 + \partial_{\alpha} \zeta_3 \int_{-1}^{x_3} f_{\alpha} - \int_{-1}^{x_3} f_3 \\ &\quad + x_3 \int_{-1}^{x_3} \partial_{\alpha} f_{\alpha} - \int_{-1}^{x_3} t \partial_{\alpha} f_{\alpha} + K(x_1, x_2). \end{aligned}$$

$K(x_1, x_2)$  is calculated, due to (9), (55) and (61) :

$$\text{for } x_3 = 1, \text{ we have : } K(x_1, x_2) = \frac{3}{4} m_{\alpha\beta} \partial_{\alpha\beta} \zeta_3 + \frac{1}{2} f_3^0 - \frac{1}{4} \partial_{\alpha} f_{\alpha}^0 - \frac{1}{2} \partial_{\alpha} \zeta_3 f_{\alpha}^0 \\ + \frac{1}{2} \partial_{\alpha} f_{\alpha}^1 + \frac{1}{2} \chi(\omega_{\beta}) (-\tilde{\Sigma}_{33}^+ \tilde{n}_3|_{\tilde{\omega}_{\beta}^+} + \tilde{\Sigma}_{33}^- \tilde{n}_3|_{\tilde{\omega}_{\beta}^-}),$$

for  $x_3 = -1$ , the same expression is found.

Finally, the function  $\Sigma_{33}$  can be written as presented in the conclusion.

## 6. CONCLUSION :

We shall gather all the results in the following theorem.

**Theorem** : Let  $(\tilde{\Sigma}, \Sigma, \tilde{u}, u)$  be the first term of the asymptotic expansion of a solution  $(\tilde{\Sigma}(\epsilon), \Sigma(\epsilon), \tilde{u}(\epsilon), u(\epsilon))$  of the problem  $\mathcal{P}_{\epsilon}$ , in correspondence with the unknowns  $(\tilde{\Sigma}^{\epsilon}, \Sigma^{\epsilon}, \tilde{u}^{\epsilon}, u^{\epsilon})$  by the scalings (15) and (16). Let us recall the assumptions (18), (19), (20) and (33). Then, we have :

$$(43) \quad \tilde{\Sigma}_{ij} = 2\tilde{\mu} \tilde{e}_{ij}(\tilde{u}) + \zeta_{ij} \tilde{\lambda} \tilde{e}_{pp}(\tilde{u}) \text{ in } \tilde{\Omega},$$

and :

$$(54) \quad \begin{cases} -\partial_j \tilde{\Sigma}_{ij} = \tilde{f}_i & \text{in } \tilde{\Omega} \\ \tilde{\Sigma}_{ij} \cdot \tilde{n}_j = 0 & \text{on } \partial\tilde{\Omega} \setminus \partial\tilde{\omega}_{\beta} \end{cases}$$

$$(42) \quad \Sigma_{\alpha\beta} = 2\mu \{ [e_{\alpha\beta}(u) + \frac{1}{2} \partial_{\alpha} u_3 \partial_{\beta} u_3] + \frac{\lambda}{\lambda+2\mu} [\partial_{\mu} u_{\mu} + \frac{1}{2} \partial_{\mu} u_3 \partial_{\mu} u_3] \delta_{\alpha\beta} \}.$$

Moreover, the displacement  $u$  is a Kirchhoff-Love field, i.e. there exist  $\zeta_{\alpha} \in W^{1,p}(\omega)$  and  $\zeta_3 \in W^{2,p}(\omega)$  such that :

$$(34) \quad \begin{cases} u_{\alpha} = \zeta_{\alpha} - x_3 \partial_{\alpha} \zeta_3, \\ u_3 = \zeta_3, \end{cases} \text{ in } \tilde{\Omega},$$

and these functions are solutions of the following problems :

$$\begin{aligned} & -\partial_{\alpha\beta} m_{\alpha\beta} - \partial_{\beta} (n_{\alpha\beta} \partial_{\alpha} \zeta_3) = f_3^0 + \partial_{\alpha} f_{\alpha}^1 - \chi(\omega_{\beta}) (\tilde{\Sigma}_{33}^+ \tilde{n}_3|_{\tilde{\omega}_{\beta}^+} + \tilde{\Sigma}_{33}^- \tilde{n}_3|_{\tilde{\omega}_{\beta}^-}) \text{ in } \omega, \\ (55) \quad & \begin{cases} m_{\alpha\beta} \nu_{\alpha} \nu_{\beta} = 0, \quad m_{\alpha\beta} \nu_{\alpha} \tau_{\beta} = 0 & \text{on } \gamma_0, \\ -(\partial_{\alpha} m_{\alpha\beta} + n_{\alpha\beta} \partial_{\alpha} \zeta_3) \nu_{\beta} = f_{\alpha}^1 \nu_{\alpha} & \text{on } \gamma_0, \\ \zeta_3 = \partial_{\nu} \zeta_3 = 0 & \text{on } \gamma_0, \end{cases} \\ (56) \quad & \begin{cases} -\partial_{\alpha} n_{\alpha\beta} = f_{\beta}^0 & \text{in } \omega, \\ n_{\alpha\beta} \nu_{\beta} = 0 & \text{on } \gamma_1, \\ \zeta_{\alpha} = 0 & \text{on } \gamma_0, \end{cases} \end{aligned}$$

where  $m_{\alpha\beta}$  and  $n_{\alpha\beta}$  are given by (48) and (49),  $f_i^0$  and  $f_i^1$  by (50) and (51).

Moreover, we have :

$$(63) \quad \Sigma_{\alpha 3} = \frac{3}{4} (1-x_3^2) \partial_{\beta} m_{\alpha\beta} + \frac{1}{2} f_{\alpha}^0 (x_3+1) - \int_{-1}^{x_3} f_{\alpha} \quad \text{in } \Omega ,$$

$$(64) \quad \begin{aligned} \Sigma_{33} = & - \frac{3}{4} (x_3 - \frac{1}{3} x_3^3) \partial_{\alpha\beta} m_{\alpha\beta} - \frac{3}{4} (x_3^2 - 1) m_{\alpha\beta} \partial_{\alpha\beta} \zeta_3 - x_3 \partial_{\beta} (\frac{1}{2} \partial_{\alpha} \zeta_3 n_{\alpha\beta}) \\ & - \frac{1}{4} \partial_{\alpha} f_{\alpha}^0 (x_3+1)^2 - \frac{1}{2} \partial_{\alpha} \zeta_3 f_{\alpha}^0 (x_3+1) + \partial_{\alpha} \zeta_3 (\int_{-1}^{x_3} f_{\alpha}) \\ & - \int_{-1}^{x_3} f_3 + \frac{1}{2} f_3^0 + x_3 \int_{-1}^{x_3} \partial_{\alpha} f_{\alpha} + \frac{1}{2} \partial_{\alpha} f_{\alpha}^1 - \int_{-1}^{x_3} t \partial_{\alpha} f_{\alpha} \\ & + \frac{1}{2} \chi (\omega_{\beta}) (- \tilde{\Sigma}_{33}^+ \tilde{n}_3 |_{\tilde{\omega}_{\beta}^+} + \tilde{\Sigma}_{33}^- \tilde{n}_3 |_{\tilde{\omega}_{\beta}^-} ) \quad \text{in } \Omega . \end{aligned}$$

Let us recall that  $\tilde{\lambda}$ ,  $\tilde{\mu}$ ,  $\lambda$  and  $\mu$  are given by (5) and (19).

Proof : This theorem is a direct consequence of what was proved previously.  $\square$

Let us compare equations (43), (54), (42), (34) and (56) with the limit problem presented by CIARLET, LE DRET & NZENGWA [1], established by a convergence method in the framework of linearized elasticity ; the linear part of our formulation is - fortunately - the same. The extra term :  $-\partial_{\beta} (n_{\alpha\beta} \partial_{\alpha} \zeta_3)$  that appears in (55) is not surprising as it was present in the Von-Kármán equations - see CIARLET & RABIER [13] - and the equations described by DAVET [7] for a single plate. Let us remark that the Lagrangian multiplier :

$$\chi (\omega_{\beta}) ( \tilde{\Sigma}_{33}^+ \tilde{n}_3 |_{\tilde{\omega}_{\beta}^+} + \tilde{\Sigma}_{33}^- \tilde{n}_3 |_{\tilde{\omega}_{\beta}^-} )$$

appearing in (55) is the same as in CIARLET, LE DRET & NZENGWA [1] , and we refer to this article for any comment about this term.

Let us note that  $\tilde{\Sigma}$  and  $\tilde{u}$  are linked by a linear constitutive law, (43), and that (54) has the structure of the three-dimensional linearized elasticity system; the reasons for these facts are the asymptotic orders chosen in (18) and (19), forces and densities being more important in the plate part than in the three-dimensional part.

Finally, let us remark that our method applied to the special case of Saint-Venant-Kirchhoff materials yields the same results as for general materials. Nevertheless, more general materials are amenable to John Ball's theory of existence of minimizers for the elastic energy whereas Saint-Venant-Kirchhoff materials do not.

